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On Packing Connectors

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Given an undirected graph $G=(V, E)$ and a partition $\{S, T\}$ of V, an $S-T$ connector is a set of edges $F \subseteq E$ such that every component of the subgraph (V, F) intersects both S and T. We show that G has k edge-disjoint S-T connectors if and only if $|\delta_G(V_1) \cup \cdots \cup \delta_G(V_t)| \geqslant kt$ for every collection $\{V_1, ..., V_t\}$ of disjoint non-empty subsets of S and for every such collection of subsets of T. This is a common generalization of a theorem of Tutte and Nash-Williams on disjoint spanning trees and a theorem of König on disjoint edge covers in a bipartite graph.

1. INTRODUCTION

Let $G=(V, E)$ be an undirected graph, S a subset of its vertices, and T the complement of S in V. An *S-T connector* in G is a set F of edges such that every component of the subgraph (V, F) intersects both S and T. Let k be a nonnegative integer. In this note, we prove the following theorem on packing S-T connectors.

THEOREM 1. *G contains k edge-disjoint S-T connectors if and only if $|\delta(W)| \geqslant k\,|W|$ for every subpartition W of S or T.*

A *subpartition* W of a set X is a collection of pairwise disjoint nonempty subsets of X. If $W=\{U_1, ..., U_t\}$ is a subpartition of S or T, then $\delta(W)$ denotes the set of edges with one end in U_i and one end in $V \backslash U_i$ for some index i.

Theorem 1 has two well-known special cases. First, if G is bipartite with colour classes S and T, then an S-T connector is an edge cover of G (a set of edges covering all vertices), and Theorem 1 specializes to a theorem of König [5] and Gupta [2], saying that the maximum number of edge-disjoint edge covers of a bipartite graph is equal to the minimum vertex degree. Second, if either S or T is a singleton, then an S-T connector is a connected spanning subgraph of G, and Theorem 1 specializes to a result

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of Tutte [9] and Nash-Williams [6], giving a necessary and sufficient condition for a graph to have k disjoint spanning trees. We state this result here as a lemma, since we will use it in the proof of Theorem 1.

LEMMA 1. *Let $G=(V, E)$ be an undirected graph. Then G contains k edge-disjoint spanning trees if and only if $|\delta(P)| \geqslant k(|P|-1)$ for every partition P of V into nonempty subsets.*

Lemma 1 is a special case of the matroid base packing theorem.

At this point, observe that an S-T connector is a common spanning set of two matroids on E, namely the cycle matroids of the graphs G_S and G_T, respectively. Here, G_S is the graph obtained from G by shrinking the set S into a single vertex s (if an edge of G connects two vertices in S, then in G_S there is a loop corresponding to this edge), and G_T, t are defined similarly. Therefore, matroid intersection provides a min–max relation for the minimum cardinality (or weight) of an S-T connector in G. However, no general theorem is known for the packing of common spanning sets of two matroids. Thus, our theorem gives a case where a min–max relation for packing common spanning sets of two matroids *is* possible (although graphic matroids generally axe not "strongly base orderable"). (For matroid theory we refer to [10].)

The concept of an S-T connector in an undirected graph is related to the concept of a bibranching in a directed graph. Given a directed graph $D=(V, A)$ and a set $S \subseteq V$ (with $T := V \backslash S$), an *S-T bibranching* is a set of arcs $B \subseteq A$ containing a directed $v-T$ path for every $v \in S$ and a directed $S-v$ path for every $v \in T$.

With respect to packing bibranchings, Schrijver [7] proved the following result, which is the second constituent of the proof of Theorem 1.

LEMMA 2. *Let $D=(V, A)$ be a digraph, let $S \subset V$, and let $T=V \backslash S$. Then D contains k arc-disjoint S-T bibranchings if and only if $|\delta_D^+(U)| \geqslant k$ for every nonempty $U \subseteq S$ and $|\delta_D^-(U)| \geqslant k$ for every nonempty $U \subseteq T$.*

Here, $\delta_D^+(U)$ denotes the set of arcs leaving U and $\delta_D^-(U)$ denotes the set of arcs entering U in D.

2. PACKING CONNECTORS

In this section we prove Theorem 1 by combining Lemma 1 and Lemma 2.

Proof of Theorem 1. Necessity is straightforward. To see sufficiency, let G be such that $|\delta(W)| \geqslant k\,|W|$ for every subpartition W of S or T. Then G_S satisfies the condition of Lemma 1 (if P is a partition of the vertex set of G_S, omit the class of P that contains s to obtain a subpartition W of T

with $|\delta(W)| = |\delta(P)|$ and $|W| = |P| - 1$). Therefore, it contains k disjoint spanning trees. The same holds for G_T. Now orient the edges of the spanning trees in G_S away from s and orient the edges of the spanning trees in G_T towards t. Note that there is no conflict for edges that are both in a spanning tree of G_S and in a spanning tree of G_T, since these edges connect S and T. Orienting the remaining edges of G arbitrarily, we obtain an orientation D of G. Clearly, $|\delta_D^-(U)| \geqslant k$ for every $U \subseteq T$ and $|\delta_D^+(U)| \geqslant k$ for every $U \subseteq S$. Therefore, by Lemma 2 D contains k arc-disjoint S-T bibranchings. Since each bibranching in D gives an S-T connector in G, this implies the theorem. ∎

The above proof gives rise to a polynomial algorithm for packing S-T connectors. Indeed, packing spanning trees can be done with any matroid partition algorithm (or alternatively, Barahona [1] reduces the problem to maximum flow computations). Moreover, disjoint bibranchings can be found in polynomial time, using the ellipsoid method (see [7]). A direct combinatorial algorithm for packing connectors is described in a subsequent paper [3]. An extension of the method used in that paper also yields a combinatorial algorithm for packing bibranchings.

For the problem of finding a minimum-weight bibranching a combinatorial algorithm is described in [4].

3. POLYHEDRAL INTERPRETATION

In this section we show that Theorem 1 implies the integer rounding property for a set of linear inequalities associated with packing S-T connectors. (For background, see [8].)

Assume that G contains an S-T connector. Equivalently, both G_S and G_T are connected. Because an S-T-connector is a common spanning set of two matroids, the convex hull of all incidence vectors of S-T connectors in G can be derived from the theory of matroid polytopes:

$$\text{conv.hull}\{\chi^F \mid F \in \mathscr{F}\} = \{x \in \mathbb{R}^E \mid 0 \leqslant x \leqslant 1,\ x(\delta(W)) \geqslant |W| \text{ for each } W \in \mathscr{W}\}.$$

Here, χ^F denotes the incidence vector of a set $F \subseteq E$, and $\mathscr{F}$ the set of all S-T connectors of G. Moreover, $\mathscr{W}$ denotes the set of all subpartitions of S and T. Finally, if $x \in \mathbb{R}^E$ and $F \subseteq E$, $x(F)$ is short for $\sum_{e \in F} x(e)$.

It follows that the polyhedra

$$P := \text{conv.hull}\{\chi^F \mid F \in \mathscr{F}\} + \mathbb{R}_+^E$$

and

$$Q := \text{conv.hull}\{\chi^{\delta(W)}/|W| \mid W \in \mathscr{W}\} + \mathbb{R}_+^E$$

form a blocking pair. In other words, $P = \{z \in \mathbb{R}_+^E \mid x^T z \geqslant 1 \; \forall x \in Q\}$ and $Q = \{x \in \mathbb{R}_+^E \mid z^T x \geqslant 1 \; \forall z \in P\}$.

Now, let M be the $\mathscr{F} \times E$ matrix with rows the incidence vectors of all S-T connectors of G. Then the fact that P and Q form a blocking pair implies:

$$\begin{aligned}\min\{w^T\chi^{\delta(W)}/|W| \mid W \in \mathscr{W}\} &= \min\{w^T x \mid x \geqslant 0,\, Mx \geqslant \mathbf{1}\}\\ &= \max\{y^T\mathbf{1} \mid y \geqslant 0,\, y^T M \leqslant w\}. \qquad (1)\end{aligned}$$

The last equality is linear programming duality.

Theorem 1 has the following polyhedral formulation:

THEOREM 2. *For every* $w: E \to \mathbb{Z}_+$

$$\max\{y^T\mathbf{1} \mid y \geqslant 0,\, y^T M \leqslant w,\, y \textit{ integral}\} = \lfloor \min\{w^T\chi^{\delta(W)}/|W| \mid W \in \mathscr{W}\} \rfloor.$$

Proof. This follows from Theorem 1 by replacing every edge e of G by $w(e)$ parallel edges. ∎

COROLLARY 1. *The set of linear inequalities* $x \geqslant 0$, $Mx \geqslant \mathbf{1}$ *has the integer rounding property. That is, for every* $w: E \to \mathbb{Z}_+$

$$\max\{y^T\mathbf{1} \mid y \geqslant 0,\, y^T M \leqslant w,\, y \textit{ integral}\} = \lfloor \max\{y^T\mathbf{1} \mid y \geqslant 0,\, y^T M \leqslant w\} \rfloor.$$

Proof. Directly from Theorem 2 with (1). ∎

Corollary 1 is equivalent to: the polyhedron P has the integer decomposition property; that is, for each k, any integer vector in $k \cdot P$ is the sum of k integer vectors in P.

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